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Journal of Mathematical Psychology 48 (2004) 9–14
Journal of Mathematical Psychology
http://www.elsevier.com/locate/jmp

Biased extensive measurement: The homogeneous case

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Received 5 February 2003; revised 15 August 2003

Abstract

In the homogeneous case of one type of objects, we prove the existence of an additive scale unique up to a positive scaling transformation without transitivity of indifference and with a property of homothetic invariance weaker than monotonicity. The representation, which is a particular case of a semiorder representation, reveals a unique positive factor $\alpha \leqslant 1$ that biases extensive structures and explains departures from these standard axioms of extensive measurement ($\alpha = 1$). We interpret α as characterizing the qualitative influence of the underlying measurement process and we show that it induces a proportional indifference threshold.

Keywords: Intransitive indifference; Interval order; Semiorder; Just noticeable difference; Measurement process; Qualitative error; Homethetic preferences; Procedural utility; Procedural invariance

1. Introduction

Following Krantz, Luce, Suppes, and Tversky (1971, chap. 3), theories of extensive measurement can be formulated as a collection of axioms about a nonempty ordering $\succ$ on a set A (of objects $x, y, z \ldots \in A$) and a binary (commutative, associative) operation $\circ$ on A that permit the construction of a scale $\varphi : A \to \mathbb{R}_{>0}$ verifying

(i) $x \succ y \Leftrightarrow \varphi(x) > \varphi(y)$,

(ii) $\varphi(x \circ y) = \varphi(x) + \varphi(y)$.

For the representational theory of measurement, φ is a *ratio-scale* and is unique up to a positive scaling transformation.

A typical interpretation in physics is the measurement of mass using an equal-arm balance. The statement "$x \succ y$" is interpreted as the empirical observation that the balance tilts in favor of object x and "$x \circ y$" is interpreted as the positioning of objects x and y in the same pan of the balance. The scale φ measures the mass of the objects. Another classical application in the physical sciences is, for instance, the measurement of length (Krantz et al., 1971, Section 3.6).

Two groups of axioms are crucial to these theories. Firstly, the ordering is assumed to be *asymmetric*: $x \succ y \Rightarrow y \not\succ x$, and *negatively transitive*: $(x \not\succ y$ and $y \not\succ z) \Rightarrow x \not\succ z$. Note that these two properties imply that the ordering is also *transitive*: $(x \succ y$ and $y \succ z) \Rightarrow x \succ z$. Secondly, the combination of the ordering and the operation is assumed to verify a form of consistency called *monotonicity*: $x \succ y \Leftrightarrow (x \circ z \succ y \circ z$ for all $z \in A)$. Note that this property, joint to the asymmetry of the ordering, imply that the operation is $\succ$*-regular*: $(x \succ y$ or $y \succ x) \Rightarrow (z \circ x \neq z \circ y$ for all $z \in A)$. If φ is a real-valued function on A verifying (i) and (ii), then all these axioms necessarily hold (because they hold for the triple $\langle \mathbb{R}, >, + \rangle$).

In this paper, we restrict ourselves to *homogeneous* structures, i.e. structures for which $mx = ny$ for some $(m, n) \in \mathbb{N}_{>0} \times \mathbb{N}_{>0}$, where nx is defined inductively by $1x = x$ and $(n+1)x = nx \circ x$. Note that this assumption (in this form or in the form in Section 3) is verified in the case of unidimensional objects that are all positively valued (e.g. masses in the physical sciences, monetary gains in the social sciences). Assuming asymmetry and transitivity (i.e. without assuming negative transitivity) and replacing monotonicity by a weaker property (*homothetic invariance*: $x \succ y \Leftrightarrow nx \succ ny$ for all $n \in \mathbb{N}_{>0}$), we show there exists a scale φ that verifies (ii) and a two-way representation (i′) more general than (i). More

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0022-2496/$ - see front matter
doi:10.1016/j.jmp.2003.11.001

precisely, we prove there exists a unique positive number $\alpha \leqslant 1$ such that

(i′) $x \succ y \Leftrightarrow \alpha\varphi(x) > \varphi(y)$.

An interpretation is the measurement of mass using a balance that is not necessarily equally armed. The statement "$x \succ y$" is interpreted as the empirical observation that the balance tilts towards x independently of the arm on which x is positioned. The interpretations of "$x \circ y$" and of the scale φ do not change and α characterizes the ratio of the length of the two arms. When α equals 1, the balance is equally armed and this approach reduces to classical extensive measurement of mass.

An interpretation in social sciences is as follows. Consider the objects to be positive amounts of money. The statement "$x \succ y$" is interpreted as "x is strictly preferred to y", the statement "($x \not\succ y$ and $y \not\succ x$)" is interpreted as "x is indifferent to y" and "$x \circ y$" is interpreted as the sum of amounts x and y. Interpreting "rational behavior" as consistency with the set of axioms, we can, for instance, model a rational individual being indifferent between €100 and €101, and between €101 and €102, while strictly preferring €102 to €100. In that case, we would have $\frac{100}{101} \geqslant \alpha > \frac{100}{102}$. Moreover, such a rational individual would not be indifferent between €1 and €2. In this manner, this model allows us to interpret an observed lack of discrimination (intransitive indifference) and a diminishing marginal utility (violation of monotonicity). The function φ extensively measures the "value" of objects and α characterizes a "bias" that influences rational choice beyond the maximization of φ. When $\alpha < 1$, the individual strictly prefers x to y if and only if the value of x is greater than the value of y multiplied by a positive factor (see Fig. 1).

Note that the intuition behind these two interpretations is somehow similar. The idea is to model the empirical or qualitative influence of the measurement process, i.e. the manner by which objects are treated by the measuring device or by the individual, through a bias that combines multiplicatively with the measurement of objects.

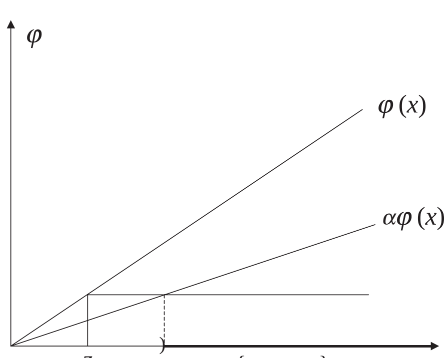

Fig. 1. A proportional bias.

In psychology, the view that insensitivity and/or inconsistency in the measurement of objects is not necessarily a nuisance but can be a source of information about the underlying processes dates back at least from Fechner in 1860 (see Suppes, Krantz, Luce, & Tversky, 1989, chap. 16). This view is related to the idea of a threshold of discrimination or just noticeable stimulus difference in psychological judgment. An important case is Weber's law of 1834 which asserts that the just noticeable difference maintains a constant ratio with respect to the intensity of the comparison stimulus. Rewriting property (i′) as $x \succ y \Leftrightarrow \varphi(x) > \varphi(y) + \frac{1-\alpha}{\alpha}\varphi(y)$, one verifies that the just noticeable difference $\Delta_\varphi(y) = \frac{1-\alpha}{\alpha}\varphi(y)$ maintains a constant ratio $c = \frac{1-\alpha}{\alpha}$ with respect to $\varphi(y)$. Weber had measured this ratio to be around $\frac{1}{20}$ when individuals measure mass without the help of a balance, which would mean α to be around $\frac{5}{6}$ in that case. In this manner, biased extensive measurement models a proportional perceptual threshold.

Insensitivity in the measurement process has been notably approached through the theory of interval orders and semiorders (Luce, 1956; Fishburn, 1985; and also Pirlot & Vincke, 1997). *Interval orders* are sets endowed with an ordering that is *irreflexive*: $x \not\succ x$, and for which $(a \succ x, b \succ y) \Rightarrow (a \succ y$ or $b \succ x)$. Fishburn (1973) provides necessary and sufficient conditions for interval orders to be represented by two real-valued functions φ and ψ with $\varphi \geqslant \psi$ such that $x \succ y \Leftrightarrow \psi(x) > \varphi(y)$. Hence, the structures represented in the present paper are interval orders with $\psi(x) = \alpha\varphi(x)$ (note that they are also *semiorders* because they verify the supplementary property $(a \succ b, b \succ c) \Rightarrow (a \succ x$ or $x \succ c)$). Instead of reflecting the insensitivity by an interval, biased extensive measurement provides for a precise measurement of the objects and of the insensitivity threshold.

In the rest of the paper, Section 2 presents the main mathematical result, Section 3 provides for a slight generalization and Section 4 concludes.

2. A homogeneous, hence denumerable, setting

We start with three primitives: a nonempty set A, a nonempty binary relation $\succ$ on A, and a closed binary operation $\circ$ on A. We write $x \sim y$ if and only if ($x \not\succ y$ and $y \not\succ x$), and $x \succsim y$ if and only if ($x \succ y$ or $x \sim y$). We note $\mathbb{N}_{>0}$ the set of positive integers, $\mathbb{Q}_{>0}$ the set of positive rational numbers and $\mathbb{R}_{>0}$ the set of positive real numbers.

Definition 1. Let A be a nonempty set, $\succ$ a nonempty binary relation on A, and $\circ$ a closed binary operation on

A. The triple $\langle A, \succ, \circ \rangle$ is called a *partially ordered homothetic structure* if the following five axioms are satisfied for all $x, y, z \in A$:

1. *Strict partial order*: $x \succ y \Rightarrow y \not\succ x$; ($x \succ y$ and $y \succ z) \Rightarrow x \succ z$.
2. *Commutativity*; *associativity*: $x \circ y = y \circ x$; $(x \circ y) \circ z = x \circ (y \circ z)$.
3. *Positivity*: $x \succ y \Rightarrow x \circ z \succ y$.
4. *Homothetic invariance*: $x \succ y \Leftrightarrow (nx \succ ny$ for all $n \in \mathbb{N}_{>0})$, where nx is defined inductively by $1x = x$ and $(n+1)x = nx \circ x$.
5. *Archimedean*: If $x \succ y$, then there exists $n \in \mathbb{N}_{>0}$ such that $nx \succ (n+1)y$.

   A nonempty set A endowed with a closed associative and commutative binary operation $\circ$ is called a *commutative semigroup*. A commutative semigroup A is in particular a $\mathbb{N}_{>0}$*-set* : $A \neq \emptyset$, and for all $x \in A$ and $m, n \in \mathbb{N}_{>0}$, we have $1x = x$ and $m(nx) = (mn)x$.

   A $\mathbb{N}_{>0}$-set A is said to be *homogeneous* if it satisfies the following condition, for all $x, y \in A$:
6. *Homogeneity*: $mx = ny$ for some $(m, n) \in \mathbb{N}_{>0} \times \mathbb{N}_{>0}$.

A commutative semigroup $\langle A, \circ \rangle$ (respectively, a $\mathbb{N}_{>0}$-set A) is said to be *regular* (resp. *homothetic-regular*) if for all $x \in A$, the map $A \to A, y \mapsto x \circ y$ (resp. the map $\mathbb{N}_{>0} \to A, m \mapsto mx$) is injective. A $\mathbb{N}_{>0}$-set A endowed with a nonempty binary relation $\succ$ is said to be *homothetic-*$\succ$*-regular* if for all $x, y \in A$, we have $(x \succ y$ or $y \succ x) \Rightarrow (nx \neq ny$ for all $n \in \mathbb{N}_{>0})$. If $\langle A, \succ, \circ \rangle$ is a partially ordered homothetic structure, then (by homothetic invariance and asymmetry), A is homothetic-$\succ$-regular. Clearly, the four notions of regularity we have introduced in this paper satisfy the following implications:

regularity $\Rightarrow$ $\succ$-regularity $\Rightarrow$ homothetic-$\succ$-regularity

and

regularity $\Rightarrow$ homothetic-regularity

$\Rightarrow$ homothetic-$\succ$-regularity.

Lemma 1. *Let $\langle A, \succ, \circ \rangle$ be a partially ordered homothetic structure. If $\langle A, \succ, \circ \rangle$ is homogeneous, then it is homothetic-regular and denumerable.*

Proof. Since $\succ$ is not empty, there exist $x, y \in A$ such that $x \succ y$ (in particular, we have $|A| \geqslant 2$). Let $z, z' \in A$, and choose $(m, n), (m', n') \in \mathbb{N}_{>0} \times \mathbb{N}_{>0}$ such that $mx = nz$ and $m'y = n'z'$ (homogeneity). By homothetic invariance, we have $m'mx \succ mm'y$, i.e. $pz \succ qz'$ with $p = m'n$ and $q = mn'$. Take $z = z'$, and suppose there exists $(a, b) \in \mathbb{N}_{>0} \times \mathbb{N}_{>0}$ such that $a > b$ and $az = bz$. Then we have $(b + k(a - b))z = bz$ for all $k \in \mathbb{N}_{>0}$, hence $m''(b + k(a - b))z = m''bz$ for all $(m'', k) \in \mathbb{N}_{>0} \times \mathbb{N}_{>0}$. Taking $m'' = q$, we can choose k big enough so that $q(b + k(a - b)) > pb$. Since $pbz \succ qbz$ (homothetic invariance), by positivity we obtain $q(b + k(a - b))z \succ qbz$, which is impossible. This implies the homothetic-regularity of A. In particular, A is an infinite set. Since A is homogeneous and homothetic-regular, for all $x, y \in A$, the set $\{\frac{m}{n} : m, n \in \mathbb{N}_{>0}, mx = ny\}$ is reduced to exactly one element, say $q_{x,y} \in \mathbb{Q}_{>0}$. For all $x \in A$, the map $A \to \mathbb{Q}_{>0}, y \mapsto q_{x,y}$ is injective. Hence A is denumerable. $\square$

We now present the main result of this paper.

Theorem 1. *Let $\langle A, \circ \rangle$ be a commutative semigroup, endowed with a nonempty binary relation $\succ$. Suppose A is homogeneous. Then the following two conditions are equivalent*:

(1) *There exist a function $\varphi : A \to \mathbb{R}_{>0}$ and a number $\alpha \in]0, 1]$ such that, for all $x, y \in A$, we have*

$$\text{(i)}' \quad x \succ y \Leftrightarrow \alpha\varphi(x) > \varphi(y),$$

$$\text{(ii)} \quad \varphi(x \circ y) = \varphi(x) + \varphi(y).$$

(2) *The triple $\langle A, \succ, \circ \rangle$ is a partially ordered homothetic structure.*

Moreover, if $\langle A, \succ, \circ \rangle$ is a partially ordered homothetic structure, then the pair (φ, α) of (1) is unique up to replacing φ by $\gamma\varphi$ for $\gamma > 0$; φ is injective; φ can be chosen with values in $\mathbb{Q}_{>0}$; and $\alpha \in \mathbb{Q}_{>0}$ if and only if there exist $x, y \in A$ such that $\alpha\varphi(x) = \varphi(y)$.

Proof. Implication (1) $\Rightarrow$ (2) is easy to prove, and left to the reader. For $x \in A$, we define the subsets of $\mathbb{Q}_{>0}$

$$\mathcal{Q}_x = \left\{\frac{m}{n} : mx \succsim nx, \exists (m, n) \in \mathbb{N}_{>0} \times \mathbb{N}_{>0}\right\},$$
$$\mathcal{P}_x = \left\{\frac{m}{n} : mx \succ nx, \exists (m, n) \in \mathbb{N}_{>0} \times \mathbb{N}_{>0}\right\}.$$

By homogeneity and homothetic invariance, for all $x, y \in A$, we have $\mathcal{Q}_x = \mathcal{Q}_y$ and $\mathcal{P}_x = \mathcal{P}_y$. So we can drop the index x in the notation $\mathcal{Q}_x$ and $\mathcal{P}_x$. By Definition 1, $\mathcal{P}$ is not empty, and by asymmetry, we have $1 \in \mathcal{Q}$. For a nonempty subset $\mathcal{X} \subset \mathbb{R}_{>0}$, let $\mathcal{X}^{-1} = \{x^{-1}, x \in \mathcal{X}\}$. We have $\mathbb{Q}_{>0} = \mathcal{Q} \cup \mathcal{P}^{-1} = \mathcal{Q}^{-1} \cup \mathcal{P}$ and $\mathcal{Q} \cap \mathcal{P}^{-1} = \mathcal{Q}^{-1} \cap \mathcal{P} = \emptyset$.

By positivity and homothetic invariance, we have $q \in \mathcal{Q} \Rightarrow \mathbb{Q}_{\geqslant q} \subset \mathcal{Q}$ and $q \in \mathcal{P} \Rightarrow \mathbb{Q}_{\geqslant q} \subset \mathcal{P}$.

We define $r = \inf_{\mathbb{R}} \mathcal{Q}$ and $s = \inf_{\mathbb{R}} \mathcal{P}$.

Because $1 \in \mathcal{Q}$, we have $0 \leqslant r \leqslant 1$. Because of positivity, we have $s \geqslant 1$.

If $r = 0$, then for all $(m, n) \in \mathbb{N}_{>0} \times \mathbb{N}_{>0}$, there exists $(m', n') \in \mathbb{N}_{>0} \times \mathbb{N}_{>0}$ such that $(m', n') \in \mathcal{Q}$ and $\frac{m'}{n'} < \frac{m}{n}$. Hence $\frac{m}{n} \in \mathcal{Q}$.Therefore $\mathcal{P} = \emptyset$, contradiction. Hence $0 < r \leqslant 1$. The same argument implies that $\mathbb{Q}_{>r} \subset \mathcal{Q}$.

Suppose $r \in \mathbb{Q} \backslash \mathcal{Q}$. Take $(m,n) \in \mathbb{N}_{>0} \times \mathbb{N}_{>0}$ such that $r = \frac{m}{n}$. Since $r \notin \mathcal{Q}$, we have $nx \succ mx$ and thus (archimedean axiom) $pnx \succ (p+1)mx$ for some $p \in \mathbb{N}_{>0}$. Therefore $\frac{(p+1)}{p} r \notin \mathcal{Q}$ which contradicts $\mathbb{Q}_{>r} \subset \mathcal{Q}$. Therefore, $r \in \mathbb{Q} \Rightarrow r \in \mathcal{Q}$.

Finally, we have $\mathcal{Q} = \mathbb{Q}_{\geqslant r}$, and also $\mathcal{P} = \mathbb{Q}_{>1/r}$. Hence, $s = \frac{1}{r}$.

Let $x \in A$, and denote by $f_x : A \to \mathbb{Q}_{>0}$ the function $y \mapsto q_{x,y}$ defined in the proof of Lemma 1. Let $y, y' \in A$. We write $mx = ny$ and $m'x = n'y'$ for some $(m,n), (m',n') \in \mathbb{N}_{>0} \times \mathbb{N}_{>0}$. Since $(n'm + nm')x = nn'(y \circ y')$, we have $q_{x,y \circ y'} = \frac{n'm+nm'}{nn'} = \frac{m}{n} + \frac{m'}{n'}$, i.e. $f_x(y \circ y') = f_x(y) + f_x(y')$. Moreover,

$$y \succ y' \Leftrightarrow n'ny \succ nn'y' \Leftrightarrow n'mx \succ nm'x \Leftrightarrow \frac{n'm}{nm'} \in \mathcal{P} \Leftrightarrow \frac{n'm}{nm'} > s$$

and

$$\frac{n'm}{nm'} > s \Leftrightarrow \frac{m}{n} > s\frac{m'}{n'} \Leftrightarrow rf_x(y) > f_x(y').$$

So we have proved that the pair $(\varphi, \alpha) = (f_x, r)$ verifies conditions (i′) and (ii) of Theorem 1. By construction φ is $\mathbb{Q}_{>0}$-valued.

Let $f' : A \to \mathbb{R}_{>0}$ be a function such that $f'(y \circ z) = f'(y) + f'(z)$ for all $y, z \in A$. Let $y \in A$, and write $mx = ny$ for some $(m,n) \in \mathbb{N}_{>0} \times \mathbb{N}_{>0}$. Then we have $mf'(x) = f'(mx) = f'(ny) = nf'(y)$, i.e. $f'(y) = \gamma f_x(y)$ with $\gamma = f'(x)$. Then φ is unique up to a positive scaling transformation. By homothetic invariance, this implies the uniqueness of α: suppose there exists $\beta \in]0,1]$ with $\beta \neq \alpha$, such that $x \succ y \Leftrightarrow \beta\varphi(x) > \varphi(y)$. We can assume that $\alpha < \beta$. Let $y, z \in A$ such that $y \succ z$. There exists $(m,n) \in \mathbb{N}_{>0} \times \mathbb{N}_{>0}$ such that $\beta > \frac{m}{n} \frac{\varphi(y)}{\varphi(z)} \geqslant \alpha$. So we have $\beta\varphi(nz) > \varphi(my)$ and $\alpha\varphi(nz) \geqslant \varphi(my)$, that is $nz \succ my$ and $nz \not\succ my$, contradiction. Let us prove that φ is injective: let $x, y \in A$ such that $\varphi(x) = \varphi(y)$. By homogeneity, there exist $m, n \in \mathbb{N}_{>0}$ such that $mx = ny$. From condition (ii), we have $m\varphi(x) = n\varphi(y)$, which implies $m = n$; hence $x = y$ by homothetic-regularity (Lemma 1). The last assertion of the theorem follows directly from the definition of f_x. □

Corollary 1. *Let $\langle A, \succ, \circ \rangle$ be a partially ordered homothetic structure. If A is homogeneous, then the semigroup $\langle A, \circ \rangle$ is regular.*

Proof. Let $(\varphi : A \to \mathbb{R}_{>0}, \alpha \in]0,1])$ be a pair verifying conditions (i′) and (ii) of (1). Since φ is injective, the regularity of $\langle A, \circ \rangle$ is implied by condition (ii). □

Remark 1. In Theorem 1, implication (1) ⇒ (2) is true without assuming A is homogeneous. Moreover, let $\langle A, \circ \rangle$ be a commutative semigroup, and $\succ$ be a nonempty binary relation on A. If the triple $\langle A, \succ, \circ \rangle$ verifies (1), then $\succ$ is a semiorder.

Remark 2. If $\alpha = 1$, negative transitivity and monotonicity hold and the triple $\langle A, \succ, \circ \rangle$ is a *closed positive extensive structure* (Krantz et al., 1970, p. 73, Definition). So in the homogeneous case, we recover the theory of extensive measurement where (i) and (ii) are satisfied.

3. A nondenumerable generalization

Retaining the algebraic approach, we now introduce a slight modification of the setting which allows us to treat a nondenumerable (but homogeneous in some sense) set of objects or stimuli. This case would cover, for instance, a set A of objects for which we would have $\varphi(x) = 1$ and $\varphi(y) = \pi$ for some $x, y \in A$ and some irrational number π.

Let $R \subset \mathbb{R}_{>0}$ be a subset containing 1 such that for all $\lambda, \mu \in R$, we have $\lambda + \mu \in R$, $\lambda\mu \in R$, and $\lambda > \mu \Rightarrow \lambda - \mu \in R$. Since $1 \in R$, we have $\mathbb{N}_{>0} \subset R$. A nonempty set A is called a *R-set* if it is endowed with a closed operation $R \times A \to A, (\lambda, x) \mapsto \lambda \cdot x$ such that for all $x \in A$ and $\lambda, \mu \in R$, we have $1 \cdot x = x$ and $\lambda \cdot (\mu x) = (\lambda\mu) \cdot x$. A R-set A is said to be *R-regular* if for all $x \in A$, the map $R \to A, \lambda \mapsto \lambda \cdot x$ is injective.

Definition 2. Let $\langle A, \cdot \rangle$ be a R-set, and $\succ$ a nonempty binary relation on A. The triple $\langle A, \succ, \cdot \rangle$ is called a *partially ordered homotheticR-set* if the following four axioms are satisfied, for all $x, y \in A$:

1. Strict partial order (Definition 1, Axiom 1).
2. R-positivity: $x \succ y \Rightarrow (\lambda \cdot x \succ \mu \cdot y$ for all $\lambda, \mu \in R$ such that $\lambda > \mu)$.
3. R-homothetic invariance: $x \succ y \Leftrightarrow (\lambda \cdot x \succ \lambda \cdot y$ for all $\lambda \in R)$.
4. R-Archimedean: If $x \succ y$, then there exist $\lambda, \mu \in R$ with $\lambda < \mu$, such that $\lambda \cdot x \succ \mu \cdot y$.

   A R-set A is said to be *R-homogeneous* if it satisfies the following condition, for all $x, y \in A$:
5. R-homogeneity: $\lambda \cdot x = \mu \cdot y$ for some $(\lambda, \mu) \in R \times R$.

Let $F(R) \subset \mathbb{R}_{>0}$ be the subset defined by $F(R) = \{\frac{\lambda}{\mu} : \lambda, \mu \in R\}$. Since $\mathbb{N}_{>0} \subset R$ we have the inclusion $\mathbb{Q}_{>0} \subset F(R)$.

Lemma 2. *Let $\langle A, \succ, \cdot \rangle$ be a partially ordered homothetic R-set. If A is R-homogeneous, then it is R-regular.*

Proof. Assume A is R-homogeneous, and suppose there exist $z \in A$ and $(a,b) \in R \times R$ such that $a > b$ and $a \cdot z = b \cdot z$. Let $B = \{m'z : m' \in \mathbb{N}_{>0}\} \subset A$, and let $\circ$ be the closed binary operation on B defined by $(m' \cdot z) \circ (n' \cdot z) = (m' + n') \cdot z$ $(m', n' \in \mathbb{N}_{>0})$. Then, $\langle B, \circ \rangle$ is a commutative semigroup, and we have $mz = m \cdot z$ $(m \in \mathbb{N}_{>0})$. Since $\succ$ is not empty, by R-homogeneity and

R-homothetic invariance, there exist $\lambda, \mu \in R$ such that $\lambda \cdot z \succ \mu \cdot z$. By R-homogeneity and R-positivity, we have $\lambda > \mu$. Let $(m,n) \in \mathbb{N}_{>0} \times \mathbb{N}_{>0}$ such that $\frac{m}{n} > \frac{\lambda}{\mu} > 1$. By R-positivity, we have $(m\mu)\lambda \cdot z \succ (n\lambda)\mu \cdot z$, hence ($R$-homogeneity) $mz \succ nz$. In particular, $\succ$ induces by restriction a nonempty binary relation on B.

For $p \in \mathbb{N}_{>0}$, by $\mathbb{N}_{>0}$-positivity, we have $(m+p)mz \succ (mn)z$, thus (homothetic invariance) $mz \circ pz \succ nz$. Now let $\lambda', \mu' \in R$ with $\lambda' < \mu'$, such that $\lambda' \cdot mz \succ \mu' \cdot nz$ (R-Archimedean). There exists $p \in \mathbb{N}_{>0}$ such that $\frac{\lambda'}{\mu'} < \frac{p}{p+1} < 1$. By R-positivity, we have $p\mu' \cdot (\lambda' \cdot mz) \succ (p+1)\lambda' \cdot (\mu' \cdot nz)$, which implies ($R$-homothetic invariance) $pmz \succ (p+1)nz$. So we have proved that the triple $\langle B, \succ, \circ \rangle$ is a partially ordered homothetic structure (Definition 1). From Theorem 1, there exists a (unique) $\alpha \in]0,1]$, such that for all $m', n' \in \mathbb{N}_{>0}$, we have $m'z \succ n'z \Leftrightarrow \alpha m' > n$.

Let $k \in \mathbb{N}_{>0}$ such that $(\frac{a}{b})^k > \alpha^{-1}$. Since $a \cdot z = b \cdot z$, we have $a^k \cdot z = b^k \cdot z$. Let $(p,q) \in \mathbb{N}_{>0} \times \mathbb{N}_{>0}$ such that $(\frac{a}{b})^k > \frac{p}{q} > \alpha^{-1}$. Since $\alpha p > q$, we have $pz \succ qz$. By R-homogeneity and R-positivity, we obtain $a^k \cdot z \succ b^k \cdot z$; contradiction. Hence A is R-regular. □

Theorem 2. *Let $\langle A, \cdot \rangle$ be a R-set, endowed with a nonempty binary relation $\succ$. Suppose A is homogeneous. Then the two following conditions are equivalent*:

(1) *There exist a function $\varphi : A \to \mathbb{R}_{>0}$ and a number $\alpha \in]0,1]$ such that, for all $x, y \in A$ and $\lambda \in R$, we have*
 (i)′ $x \succ y \Leftrightarrow \alpha\varphi(x) > \varphi(y)$,
 (ii)′ $\varphi(\lambda \cdot x) = \lambda\varphi(x)$.

(2) *The triple $\langle A, \succ, \cdot \rangle$ is a partially ordered homothetic R-set.*

Moreover, if $\langle A, \succ, \cdot \rangle$ is a partially ordered homothetic R-set, then the pair (φ, α) of (1) is unique up to replacing φ by $\gamma\varphi$ for $\gamma > 0$; φ is injective; φ can be chosen with values in $F(R)$; and $\alpha \in F(R)$ if and only if there exist $x, y \in A$ such that $\alpha\varphi(x) = \varphi(y)$.

Proof. Roughly speaking, it suffices to replace $\mathbb{N}_{>0}$ by R and $\mathbb{Q}_{>0}$ by $F(R)$ in the proof of Theorem 1. For $x \in A$, we define the (nonempty) subsets of $F(R)$

$$\mathcal{Q}_x = \left\{ \frac{\lambda}{\mu} : \lambda \cdot x \succsim \mu \cdot x, \exists (\lambda, \mu) \in R \times R \right\},$$

$$\mathcal{P}_x = \left\{ \frac{\lambda}{\mu} : \lambda \cdot x \succ \mu \cdot x, \exists (\lambda, \mu) \in R \times R \right\}.$$

By R-homogeneity and R-homothetic invariance, we can drop the index x in the notation $\mathcal{Q}_x$ and $\mathcal{P}_x$. We have $F(R) = \mathcal{Q} \cup \mathcal{P}^{-1} = \mathcal{Q}^{-1} \cup \mathcal{P}$ and $\mathcal{Q} \cap \mathcal{P}^{-1} = \mathcal{Q}^{-1} \cap \mathcal{P} = \emptyset$. By R-positivity and R-homothetic invariance, we have $q \in \mathcal{Q} \Rightarrow F(R)_{\geqslant q} \subset \mathcal{Q}$ and $q \in \mathcal{P} \Rightarrow F(R)_{\geqslant q} \subset \mathcal{P}$. We define $s = \inf_{\mathbb{R}} \mathcal{P}$ and $r = \inf_{\mathbb{R}} \mathcal{Q}$. Because $1 \in \mathcal{Q}$, we have $0 \leqslant r \leqslant 1$, and because $\succ$ is nonempty, we have $r > 0$ and $F(R)_{>r} \subset \mathcal{Q}$. This last inclusion, joint to the R-Archimedean axiom, implies that if $r \in F(R)$, then $r \in \mathcal{Q}$. So we have $\mathcal{Q} = F(R)_{\geqslant r}$, $\mathcal{P} = F(R)_{>r^{-1}}$ and $s = r^{-1}$.

By R-regularity, for all $x, y \in A$, there exists a unique $q_{x,y} \in F(R)$ such that $\{\frac{\lambda}{\mu} : \lambda, \mu \in R, \lambda \cdot x = \mu \cdot y\} = \{q_{x,y}\}$; and for all $\lambda, \mu \in R$, we have $q_{\lambda \cdot x, \mu \cdot y} = \frac{\mu}{\lambda} q_{x,y}$. Let $x \in A$. We define a function $f_x : A \to F(R)$ by $f_x(y) = q_{x,y}$. As in the proof of Theorem 1, we verify that the pair $(\varphi, \alpha) = (f_x, r)$ verifies the conditions (i′) of (1); and condition (ii′) is satisfied by construction. All the remaining assertions of Theorem 2 are obtained as in the proof of Theorem 1. □

Corollary 2. *Let $\langle A, \succ, \cdot \rangle$ be a partially ordered homothetic $\mathbb{N}_{>0}$-set. Suppose A is homogeneous. Then there exists a unique closed binary operation $\circ$ on A extending the structure of $N_{>0}$-set, which makes the triple $\langle A, \succ, \circ \rangle$ a partially ordered homothetic structure.*

Proof. Let $(\varphi : A \to \mathbb{R}_{>0}, \alpha \in]0,1])$ be a pair verifying conditions (i′) and (ii′) of (1) in Theorem 2. Since φ is injective, we can define a closed binary operation $\circ$ on A by $x \circ y = \varphi^{-1}(\varphi(x) + \varphi(y))$. By construction, $\langle A, \circ \rangle$ is a commutative semigroup, $mx = m \cdot x$ $(x \in A, m \in \mathbb{N}_{>0})$, and φ satisfies the condition (ii) of (1) in Theorem 1. Hence by Theorem 1, the triple $\langle A, \succ, \circ \rangle$ is a partially ordered homothetic structure. Now let $x \in A$. For $y, y' \in A$, there exist $(m,n) \in \mathbb{N}_{>0} \times \mathbb{N}_{>0}$ and $(m', n') \in \mathbb{N}_{>0} \times \mathbb{N}_{>0}$ such that $my = nx$ and $my' = n'x$; so we have $\varphi(y + y') = \frac{1}{mm'}\varphi(mm'(y \circ y')) = \frac{1}{mm'}\varphi(m'n + mn')x) = (\frac{n}{m} + \frac{n'}{m'})\varphi(x)$, that is $y \circ y' = (\frac{n}{m} + \frac{n'}{m'})x$. From this and the uniqueness property in Theorem 1, we obtain the uniqueness property in the corollary. □

Remark 3. In Theorem 2, implication (1) ⇒ (2) is true even if A is not R-homogeneous. On the other hand, the proof of Lemma 2 implies the following reciprocal assertion of Corollary 2: let $\langle A, \succ, \circ \rangle$ be a partially ordered homothetic structure. If A is homogeneous, then the triple $\langle A, \succ, \cdot \rangle$ defined by $m \cdot x = mx$ $(m \in \mathbb{N}_{>0})$, is a partially ordered homothetic $\mathbb{N}_{>0}$-set.

Let us finish this section by some remarks. The Corollary 2 says in particular that Theorem 2 is stronger than Theorem 1, and can be viewed as a mathematical generalization of it. But this is not the principal motivation of this section: Theorem 1 is stated in the classical setting of extensive measurement, while Theorem 2 stresses the underlying homothetic structure. Replacing condition (ii) by condition (ii′), we would like to show how the semigroup structure is secondary with respect to the $\mathbb{N}_{>0}$-structure (Corollary 2). In fact, this is the key-point in the generalization of our results to nonhomogeneous structures: actually, we are able to prove a version of Theorem 2 for nonhomogeneous A, which "contains" Theorem 1 as a particular case. Details will appear in a future work.

4. Conclusion

In this paper, we have axiomatized a class of algebraic structures $\langle A, \succ, \circ \rangle$ (resp. $\langle A, \succ, \cdot \rangle$) for which there exist a ratio scale φ and a unique positive factor $\alpha \leqslant 1$ such that $x \succ y \Leftrightarrow \alpha\varphi(x) > \varphi(y)$ and $\varphi(x \circ y) = \varphi(x) + \varphi(y)$ (resp. $\varphi(\lambda \cdot x) = \lambda\varphi(x)$). The factor α is said to "bias" the extensive measurement of the objects of the set A and we have called such "twisted" representations: "biased extensive measurement". The results are obtained using a structural assumption called "homogeneity" that covers, with its variant R-homogeneity, the case of unidimensional objects that are all positively valued.

This approach maintains a precise measurement of the objects while reflecting a form of insensitivity (the symmetric part of the ordering is transitive if and only if $\alpha = 1$) and a form of inconsistency (monotonicity holds similarly) in the measurement process. Hence, the bias α can be said to characterize a form of qualitative or procedural error. An example in the physical sciences and an example in the social sciences have been illustrating the main result.

Besides generalizing the result for nonhomogeneous structures, future developments shall link this approach with probabilities and with the notions of procedural invariance and procedural utility, which were the initial motivation of this work (Le Menestrel & Van Wassenhove, 2001) but shall be given more specific formulation and discussion.

Acknowledgments

We thank Manuel Baucells, Enrico Decidue, Peter Fishburn, Jean-Yves Jaffray, Christoph Loch, Duncan Luce, Patrick Suppes, anonymous referees and the acting editor Jean-Paul Doignon for their comments on earlier drafts of this paper.

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