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MARC LE MENESTREL and BERTRAND LEMAIRE

# RATIO-SCALE MEASUREMENT WITH INTRANSITIVITY OR INCOMPLETENESS: THE HOMOGENEOUS CASE

ABSTRACT. In the homogeneous case of one-dimensional objects, we show that any relation that is positive and homothetic can be represented by a ratio-scale and a unique and constant biasing factor. This factor may favor or disfavor the preference for an object over another. In the first case, preferences are complete but not transitive and an object may be preferred even when its value is lower. In the second case, preferences are asymmetric and transitive but not negatively transitive and it may not be sufficient for an object to have a greater value to be preferred. In this manner, the biasing factor reflects the extent to which preferences may depart from a maximization process.

KEY WORDS: intransitive preferences, incomplete preferences, irrational behavior, emotional behavior, procedural concerns, ethical values, biased measurement, scale-invariance, homotheticity.

## 1. INTRODUCTION

To which extent can we measure objects when empirical observation is inconsistent or incomplete? To which extent can such phenomena be explained with a "biasing factor"? Also, could such a factor be "measured"?

In the homogeneous case of one-dimensional objects, this paper identifies conditions under which there exists a quantitative measurement of objects even though the primitive relation may be intransitive or incomplete. Moreover, it shows that the departure from these standard axioms of measurement can be characterized by a unique and constant biasing factor. Given an ordering $\succ$ on a set $A$ (of objects $x, y, \dots, \in A$), the main representation theorem proves the existence of a positively-valued function $\varphi : A \to \mathbb{R}_{>0}$ that is unique up to multiplication by a positive scalar (i.e. a ratio scale) and a unique biasing factor $\beta \in [0, 1]$ such that

$$x \succ y \Leftrightarrow \beta\varphi(x) > (1-\beta)\varphi(y). \qquad (i)$$

Consider for instance the interpretation that "$x \succ y$" reflects the preference for $x$ over $y$. Then, $\varphi$ provides some measure of the "value" or "utility" of objects and $\beta$ provides some measure of a "factor", "bias", or "disposition" that influences preferences in a specific situation. Naturally, if $\beta = \frac{1}{2}$, preferences are represented by the maximization of $\varphi$. But if $\beta > \frac{1}{2}$, then the biasing factor "favors" the preference for $x$, and $x$ may be preferred even if its value $\varphi(x)$ is lower than $\varphi(y)$. If $\beta < \frac{1}{2}$ then the factor "disfavors" the preference for $x$ and it is not sufficient for $x$ to have a greater value to be preferred. If $\beta = 0$ then preferences are empty and if $\beta = 1$, they hold for any pair of objects. In these two limit cases, the biasing factor determines preferences independently of the value of objects.

Such a representation may be of interest to model preferences that appear "irrational", for instance because of a particular context or because of some factor that is difficult to control experimentally or even impossible to define conceptually. A typical example would involve procedural concerns, ethical values or emotional considerations, which are often elusive and thus kept out of the representation of preferences. For instance, we may observe individuals preferring less money than more money because of ethical reasons (honesty, equity, etc.) without being able to model exactly what these ethical reasons are. Indeed, it is often a question of interpretation to consider whether, in a specific situation, a particular action is "honest" or not. Moreover, this interpretation may be different for the individual who acts and for the observer. In the case of a contextual bias or of emotional considerations, the individual may not even be aware of the specific feature of the situation that influences preferences. Without providing for a specific interpretation explaining why some distortion is observed, the representation proposed in $(i)$ shows that preferences may be consistent with a quantitative value assigned to each object and a unique factor proper to the situation at hand.

In the homogeneous case treated in this paper, our representation ($i$) is obtained without assuming specific conditions for the primitive relation $\succ$ itself. Indeed, if the biasing factor favors the preference over $x$ (i.e. $\beta > \frac{1}{2}$) then the relation $\succ$ is complete but not transitive. If the biasing factor disfavors the preference over $x$ (i.e. $\beta < \frac{1}{2}$) then the relation $\succ$ is asymmetric and transitive, but its negation is not transitive, leading to intransitivity of indifference. In this manner, a broad class of relations are covered. However, the model is not arbitrary and two main axioms are necessary for our results. First, the relation is assumed to be *positive*. In the interpretation above, this means that, if an object $x$ is preferred to an object $y$, then any quantity $mx$ of object $x$ is preferred to any quantity $ny$ of object $y, m$ and $n$ being positive natural numbers such that $m > n$. Second, preferences verify a form of scale *invariance* called *homotheticity*. This means that $x$ is preferred over $y$ if and only if any quantity $mx$ is preferred to $my$ (again, $m$ being any positive natural number). In the homogeneous case, representations ($i$) above and ($i'$) below show that any relation that is positive and homothetic can be modeled with a ratio-scale and a unique factor.

We obtain the representation ($i$) assuming that the primitive relation, noted $\succ$ verifies an Archimedean axiom. When the primitive relation is non-Archimedean, we note it $\succsim$ and we obtain a representation of the form

$$x \succsim y \Leftrightarrow \beta\varphi(x) \geqslant (1-\beta)\varphi(y). \qquad (i')$$

In that case, we define two relations $\succ$ and $\sim$ such that $\succ$ is represented by ($i$) and $\sim$ is represented by

$$x \sim y \Leftrightarrow \beta\varphi(x) = (1-\beta)\varphi(y). \qquad (i'')$$

In this manner, we naturally have $x \succsim y \Leftrightarrow (x \succ y$ or $x \sim y)$ and the relation $\sim$ defines a particular type of "just indifference". In general, such indifference is neither symmetric nor transitive. In terms of interpretation, $x$ is "just indifferent" to

$y$ if and only if $x$ is preferred to $y$ but $mx$ is not preferred to $ny$ for any positive natural numbers $m$ and $n$ such that $m < n$.

Both representations $(i)$ and $(i')$ show that we can obtain the ratio-scale $\varphi$ such that

$$\varphi(mx) = m\varphi(x). \qquad (ii)$$

Hence, the measurement of an object increases linearly with the quantity of that object. Combined with expression $(i'')$, this feature may be useful to experimentally elicit the exact value of the biasing factor.

Consider for instance that, in a specific situation, it is observed that an individual prefers to earn an amount of 10 euros rather than an amount of 15 euros. According to representation $(i')$, this means that a biasing factor $\beta \geqslant \frac{\varphi(15euros)}{\varphi(10euros)+\varphi(15euros)}$ favors the preference for the 10 euros. Using Equation $(ii)$ we have $\varphi(15euros) = 15\varphi(1euro)$ and $\varphi(10euros) = 10\varphi(1euro)$. Therefore, $\beta \geqslant \frac{3}{5}$. Now, if we observe that, *ceteris paribus*, any increase of the 15 euros lead the individual not to prefer the amount of 10 euros, the conditions of representation $(i'')$ are met and we have $\beta = \frac{3}{5}$. Using the axioms of positivity and homotheticity, this value of $\beta$ can then be used to make predictions about preferences over different amounts of money. For instance, positivity imposes that any amount greater than 10 euros should be preferred to 15 euros and homotheticity imposes that an amount of 20 euros should be preferred to 30 euros (again ceteris paribus). Finally, note that the model does not impose that the relation is invariant by translation. In our example, if we add 20 euros to both amounts, then we should observe that 30 euros are not preferred to 35 euros.

Although we have tried to sketch applications of these representations, there remains work to clarify their practical relevance. In this paper, we focus on the derivation of the mathematical results, pursuing our effort to identify general axiomatic conditions under which a ratio-scale exists, albeit not directly observed. In this respect, the results of this paper extend the results of Le Menestrel and Lemaire (2004) which, in the homogeneous case, show that a ratio-scale exists without transitivity of indifference. Assuming more restrictive axioms than the present paper does (notably assuming that $\succ$ is asymmetric and transitive), we had obtained a representation very similar to ($i$) but covering only the case where $0 < \frac{\beta}{(1-\beta)} \leqslant 1$. Hence, we could not model a factor that favors a preference for an object. In Lemaire and Le Menestrel (2006) and in Le Menestrel and Lemaire (2006), we have generalized these former results to the non-homogeneous case, covering sets of objects which are not necessarily unidimensional. Then, the biasing factor is not necessarily constant. Similarly, we are trying to generalize the results presented here to non-homogeneous sets.

## 2. EXTENDING THE HOMOGENEOUS CASE OF BIASED MEASUREMENT

Let $A$ be a nonempty set of elements $x, y, z \ldots \in A$. Denote $\mathbb{N}^*$ the set of positive integers, and assume $A$ to be endowed with a map $\mathbb{N}^* \times A \to A, (m, x) \mapsto mx$ such that $(mm')x = m(m'x)$ and $1x = x$. such a $A$ is called a $\mathbb{N}^* - set$. Note that the results we obtain for $\mathbb{N}^*$-sets are true (mutatis mutandis) for $\mathbb{R}_{>0}$-sets, where $\mathbb{R}_{>0}$ denotes the set of positive real numbers. Hence, the main results presented here (Theorems 1 and 2) would remain valid if one wants to let $n$ be a positive real number instead of a positive natural number (here, we use natural numbers because we do not need quantities to be non-denumerable to obtain our results).

We say that $A$ is *homogeneous* if, for all $x, y \in A$, there exists $(m, n) \in \mathbb{N}^* \times \mathbb{N}^*$ such that $mx = ny$. A homogeneous set can hence be readily interpreted as consisting of quantities of one-dimensional objects. We note $\mathbb{Q}_{>0}$ the set of positive rational numbers.

Let $\succ$ be a binary relation on $A$ and consider the three following axioms $(x, y \in A; m, n \in \mathbb{N}^*)$:

**A1** (*Positivity*): $\forall (x, y, m, n)$ such that $m > n$, we have $x \succ y \Rightarrow mx \succ ny$.

**A2** (*Homotheticity*): $\forall (x, y, m)$ we have $x \succ y \Leftrightarrow mx \succ my$;

**A3** (*Archimedean*): $\forall(x, y)$ such that $x \succ y, \exists(m, n)$ such that $m < n$ and $mx \succ ny$.

THEOREM 1. *Let $A$ be a $\mathbb{N}^*$ – set endowed with a binary relation $\succ$ that verifies A1, A2 and A3. Suppose $A$ is homogeneous. Then there exist a function $\varphi : A \to \mathbb{R}_{>0}$ and a number $0 \leqslant \beta \leqslant 1$ such that, for all $x, y \in A$ and $m \in \mathbb{N}^*$, we have*

$$x \succ y \Longleftrightarrow \beta\varphi(x) > (1-\beta)\varphi(y), \tag{i}$$

$$\varphi(mx) = m\varphi(x) \tag{ii}$$

*Moreover, the pair $(\varphi, \beta)$ of (i) is unique up to replacing $\varphi$ by $\lambda\varphi$ for $\lambda > 0$.*

*Proof.* We can always choose a function $\varphi : A \to \mathbb{R}_{>0}$ such that $\varphi(mx) = m\varphi(x)$ for all $(x, m) \in A \times \mathbb{N}^*$. Since $A$ is homogeneous, such a function exists and is unique up to multiplication by a positive scalar; in other words, given an element $a \in A, \varphi$ is uniquely determined by its value at $a$. If the relation $\succ$ (respectively $\not\succ$) is empty, we take $\beta = 0$ (resp. $\beta = 1$). So in both cases, we have

$$x \succ y \Longleftrightarrow \beta\varphi(x) > (1-\beta)\varphi(y).$$

From now on, suppose that both the relation $\succ$ and the relation $\not\succ$ are nonempty. For $x, y \in A$, we define the subset of $\mathbb{Q}_{>0}$

$$\mathcal{P}_{x,y} = \left\{\frac{m}{n} : (m, n) \in \mathbb{N}^* \times \mathbb{N}^*, mx \succ ny\right\}.$$

Let $x, y \in A$. By A1 and A2, if $q \in \mathcal{P}_{x,y}$ then $\mathbb{Q}_{\geqslant q} \subset \mathcal{P}_{x,y}$. And since $\succ$ is nonempty and $A$ is homogeneous, we have $\mathcal{P}_{x,y} \neq \emptyset$. Put $t_{x,y} = \inf_{\mathbb{R}_{\geqslant 0}} \mathcal{P}_{x,y}$. By A1 and A2, we have $\mathbb{Q}_{>t} \subset \mathcal{P}_{x,y}$ with $t = t_{x,y}$.

Now, if $q \in \mathcal{P}_{x,y}$, then by A2 and A3, there exists $q' \in \mathbb{Q}_{<q} \cap \mathcal{P}_{x,y}$, which implies $q > t_{x,y}$. Hence, we have $\mathcal{P}_{x,y} \subset \mathbb{Q}_{>t}$ and then $\mathcal{P}_{x,y} = \mathbb{Q}_{>t}$.

We thus have

$$x \succ y \Longleftrightarrow 1 \in \mathcal{P}_{x,y} \Longleftrightarrow t_{x,y} < 1.$$

Since $\mathcal{P}_{mx,ny} = \frac{n}{m}\mathcal{P}_{x,y}$, we have $t_{mx,ny} = \frac{n}{m}t_{x,y}$. Using the homogeneity of $A$, we obtain $t_{x,y} > 0$ (recall the relation $\not\succ$ is supposed to be nonempty). Now, choose $a \in A$. Since we can always replace $\varphi$ by $\lambda\varphi$ with $\lambda = \varphi(a)^{-1}t_{a,a}$, we can suppose $\varphi(a) = t_{a,a}$. Since $t_{a,nx} = nt_{a,x}$, we thus have $\varphi(x) = t_{a,x} \in \mathbb{R}_{>0}$. Also put

$$\sigma(x, y) = t_{a,x}^{-1} t_{x,y}^{-1} t_{a,y} \in \mathbb{R}_{>0}.$$

Since $\mathcal{P}_{mx,ny} = \frac{n}{m}\mathcal{P}_{x,y}$, we have $t_{mx,ny} = \frac{n}{m}t_{x,y}$. Hence $\varphi(mx) = m\varphi(x)$ and

$$\sigma(mx, ny) = (mt_{a,y})^{-1}(\frac{n}{m}t_{x,y})^{-1} nt_{a,y} = \sigma(x, y).$$

Since $A$ is homogeneous, $\sigma$ is constant on $A \times A$; let $\beta = \frac{\sigma(A\times A)}{\sigma(A\times A)+1}$. We have $0 < \beta < 1$ (both $\succ$ and $\not\succ$ being nonempty).

Also

$$\begin{aligned} x \succ y &\Leftrightarrow t_{x,y}^{-1} > 1 \\ &\Leftrightarrow (t_{a,x}^{-1} t_{x,y}^{-1} t_{a,y}) t_{a,x} > t_{a,y} \\ &\Leftrightarrow \beta\varphi(x) > (1-\beta)\varphi(y). \end{aligned}$$

As for the uniqueness property (in the general case, i.e. without hypothesis on the emptiness of $\succ$ and $\not\succ$), let $(\psi, \gamma)$ be a pair such that $\psi : A \to \mathbb{R}_{>0,}\, \psi(mx) = m\psi(x), 0 \leqslant \gamma \leqslant 1,$ *and* $x \succ y \Leftrightarrow \gamma\psi(x) > (1-\gamma)\psi(y)$. Since $A$ is homogeneous, we necessarily have $\psi = \lambda\varphi$ for some $\lambda \in \mathbb{R}_{>0}$. It is then easy to deduce that $\gamma = \beta$. □

We can summarize the properties of $\succ$ in the following corollary:

COROLLARY 1. *Let $\succ$ be a binary relation on a homogeneous $\mathbb{N}^*$ – set $A$ that verifies A1, A2 and A3, and let $(\varphi, \beta)$ be a pair that verifies conditions $(i)$ and $(ii)$ of Theorem 1. The relation $\succ$ is*

- *nonempty if and only if $\beta > 0$,*
- *asymmetric and transitive if and only if $\beta \leqslant \frac{1}{2}$,*
- *complete if and only if $\beta > \frac{1}{2}$.*

Note also (with the notation of Corollary 1) that the relation $\not\succ$ is given by

$$x \not\succ y \Longleftrightarrow \beta\varphi(x) \leqslant (1-\beta)\varphi(y).$$

In particular, $\not\succ$ is nonempty if and only if $\beta < 1$.

In Theorem 1, we assume that the primitive relation $\succ$ is Archimedean. Now, starting with a binary relation $\succsim$ on $A$ that verifies A1 and A2, we define two binary relations $\succ$ and $\sim$ on $A$ as follows:

$$x \succ y \Leftrightarrow (x \succsim y \text{ and } \exists(m,n) \text{ such that } m < n \text{ and } mx \succsim ny);$$
$$x \sim y \Leftrightarrow (x \succsim y \text{ and } x \not\succ y).$$

As suggested by the notation, we have $x \succsim y \Leftrightarrow (x \succ y$ *or* $x \sim y)$. Since $\succsim$ is homothetic and positive, then $\succ$ and $\sim$ are homothetic, and $\succ$ is positive and Archimedean. Note that $\sim$ may not be symmetric (i.e. it may not verify $x \sim y \Rightarrow y \sim x$); and $\succ$ may not be asymmetric (i.e. it may not verify $x \succ y \Rightarrow y \not\succ x$). Note also that if $x \not\succ y \Leftrightarrow y \succsim x$ for all $x, y \in A$, then the relation $\sim$ is given by

$$x \sim y \Leftrightarrow (x \succsim y \text{ and } y \succsim x) \Leftrightarrow (x \not\succ y \text{ and } y \not\succ x);$$

in which case it is clearly symmetric. It then corresponds to the more traditional definition of indifference.

The relation $\sim$ is empty if and only if the relation $\succsim$ verifies A3, in which case we have $\succ = \succsim$. Hence assuming that $\sim$ is not empty amounts to assume that $\succsim$ verifies the following axiom ($x, y \in A; m, n \in \mathbb{N}^*$):

**A3′** (*Non-Archimedean*): $\exists(x, y)$ such that $x \succsim y$ and, $\forall(m, n)$ such that $m < n$, we have $mx \not\succsim ny$.

This leads us to a slightly different formulation of Theorem 1:

THEOREM 2. *Let $A$ be a $\mathbb{N}^*$ – set endowed with a binary relation $\succsim$ that verifies A1, A2 and A3′. Suppose $A$ is homogeneous. Then there exist a function $\varphi: A \to \mathbb{R}_{>0}$ and a number $0 < \beta \leqslant 1$ such that, for all $x, y \in A$ and $m \in \mathbb{N}^*$, we have*

$$x \succsim y \Longleftrightarrow \beta\varphi(x) \geqslant (1-\beta)\varphi(y), \qquad (i')$$
$$\varphi(mx) = m\varphi(x). \qquad (ii)$$

*Moreover, the pair* $(\varphi, \beta)$ *of (i′) is unique up to replacing* $\varphi$ *by* $\lambda\varphi$ *for* $\lambda > 0$.

*Proof.* Choose a function $\varphi: A \to \mathbb{R}_{>0}$ such that $\varphi(mx) = m\varphi(x)$ for all $(x, m) \in A \times \mathbb{N}^*$ (Cf. the proof of Theorem 1). If $\not\succsim$ is empty, we take $\beta = 1$; hence the pair $(\beta, \varphi)$ verifies $(i')$.

We now suppose $\not\succsim$ is not empty. Since $\succsim$ verifies A3′, it is not empty. Also, $\sim$ is not empty. Because $A$ is homogeneous, for all $(x, y) \in A \times A$, there exists $(m_0, n_0) \in \mathbb{N}^* \times \mathbb{N}^*$ such that $m_0 x \sim n_0 y$. We define $\mathcal{P}_{x,y}$ and $t_{x,y}$ as in the proof of Theorem 1. We also define the subset of $\mathbb{Q}_{>0}$

$$\mathcal{Q}_{x,y} = \left\{ \frac{m}{n} : (m, n) \in \mathbb{N}^* \times \mathbb{N}^*,\ mx \succsim ny \right\}.$$

So we have the inclusion $\mathcal{P}_{x,y} \subset \mathcal{Q}_{x,y}$. If $q \in \mathcal{Q}_{x,y}$, then by A1 and A2, we have $\mathbb{Q}_{\geqslant q} \subset \mathcal{Q}_{x,y}$; and by the definition of $\succ$, we have $\mathbb{Q}_{>q} \subset \mathcal{P}_{x,y} = \mathbb{Q}_{>t_{x,y}}$, and $q \geqslant t_{x,y}$. Since $\succ \neq \succsim$, the inclusion $\mathcal{P}_{x,y} \subset \mathcal{Q}_{x,y}$ is strict. Hence we have $\mathcal{Q}_{x,y} = \mathbb{Q}_{\geqslant t_{x,y}}$, and

$$x \succsim y \Longleftrightarrow 1 \in \mathcal{Q}_{x,y} \Longleftrightarrow t_{x,y} \leqslant 1.$$

Since $\not\succsim$ is supposed to be nonempty, we also have $t_{x,y} > 0$. From Theorem 1, there exists a $0 \leqslant \beta \leqslant 1$ such that $x \succ y \Longleftrightarrow \beta\varphi(x) > (1 - \beta)\varphi(y)$. We then have

$$x \succsim y \Longleftrightarrow \beta\varphi(x) \geqslant (1 - \beta)\varphi(y)$$

and

$$x \sim y \Longleftrightarrow \beta\varphi(x) = (1 - \beta)\varphi(y).$$

Since the relation $\succsim$ is nonempty, the relation $\succ$ is also nonempty, and we have $\beta > 0$.

Finally, if $(\psi, \gamma)$ is another pair verifying conditions $(i')$ and $(ii)$, then it verifies condition $(i)$ of Theorem 1, which implies $(\psi, \gamma) = (\lambda\varphi, \beta)$ for a positive scalar $\lambda$. □

We can summarize the properties of $\succsim$ in the following corollary:

COROLLARY 2. *Let* $\succsim$ *be a binary relation on a homogeneous* $\mathbb{N}^*$ – *set* $A$ *that verifies A1, A2 and A3′, and let* $(\varphi, \beta)$ *be* *a pair that verifies conditions* ($i'$) *and (ii) of. Theorem 2. The relation* $\succsim$ *is*

- *asymmetric if and only if* $\beta < \frac{1}{2}$,
- *transitive if and only if* $\beta \leqslant \frac{1}{2}$,
- *complete if and only if* $\beta \geqslant \frac{1}{2}$.

Note also (with the notation of Corollary 2) that the relation $\not\succsim$ is given by

$$x \not\succsim y \Longleftrightarrow \beta\varphi(x) < (1-\beta)\varphi(y).$$

In particular, $\not\succsim$ is nonempty if and only if $\beta < 1$.

The following two corollaries help to further understand the link between Theorem 1 and Theorem 2. A key condition is whether the biasing factor is a rational number.

COROLLARY 3. *Let* $\succsim$ *be a binary relation on a homogeneous* $\mathbb{N}^*$ *– set A that verifies A1, A2 and A3′, and let* $(\varphi, \beta)$ *be a pair that verifies conditions (i′) and (ii) of Theorem 2. Then* $\beta \in \mathbb{Q}_{>0}$.

*Proof.* For all $(x, y) \in A \times A$, there exists $(m_0, n_0) \in \mathbb{N}^* \times \mathbb{N}^*$ such that $m_0 x \sim n_0 y$ and we have $t_{x,y} = \frac{m_0}{n_0}$ (cf. the proof of Theorem 2). From the definition of $\beta$ (cf. the proof of Theorem 1), we conclude that $\beta \in \mathbb{Q}_{>0}$. □

COROLLARY 4. *Let* $\succ$ *be a nonempty binary relation on a homogeneous* $\mathbb{N}^*$ *– set A that verifies A1, A2 and A3, and let* $(\varphi, \beta)$ *be a pair that verifies conditions* ($i$) *and (ii) of Theorem 1. Let* $\succsim$ *be the binary relation defined by* $x \succsim y \Longleftrightarrow \beta\varphi(x) \geqslant (1-\beta)\varphi(y)$. *Then* $\succsim$ *verifies A1 and A2, and it verifies A3′ if and only if* $\beta \in \mathbb{Q}_{>0}$.

*Proof.* Clearly, the relation $\succsim$ verifies A1 and A2. And we have

$$x \succ y \Leftrightarrow (x \succsim y \text{ and } \exists(m,n) \text{ with } m < n \text{ such that } mx \succsim ny).$$

Hence if the relation $\succsim$ verifies A3′, from Corollary 3, we have $\beta \in \mathbb{Q}_{>0}$. Now suppose $\beta \in \mathbb{Q}_{>0}$. Since $\beta \in \mathbb{Q}_{>0}$ there exists $(m, n) \in \mathbb{N}^* \times \mathbb{N}^*$ such that $\beta m = (1-\beta)n$. Hence for all $x \in A$, we have $mx \succsim nx$ but $mx \not\succ nx$. So $\succ \neq \succsim$, and $\succsim$ verifies A3′. □

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*Addresses for correspondence*: Marc Le Menestrel, Universitat Pompeu Fabra, Departament d'Economia i Empresa, Ramon Trias Fargas 25–27, 08005 Barcelona, Spain. E-mail: Marc.Lemenestrel@upf.edu

Bertrand Lemaire UMR 8628 du CNRS, Université de Paris-Sud, Mathématiques (bât. 425), 91405 Orsay cedex, France. E-mail: Bertrand.Lemaire@math.u-psud.fr